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ON A CLASS OF ELASTICITY THEORY PROBLEMS WITH NON-CLASSICAL BOUNDARY CONDITIONS*

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A contact problem of interaction between a massive body containing a cavity and elastic shell reinforcements of these cavities is considered. It is replaced by some other just for a massive body with non-classical boundary conditions on the cavity boundaries that asymptotically describe the interaction in the original contact problem exactly. Error estimates are constructed for such a replacement in the norm L_2.

1. Formulation of the problem. Let a physically linear elastic body occupying a domain $V_0 \subset R^3$ in the non-deformed state, contain a cavity V. The latter is reinforced by a shell of thickness h whose external facial surface coincides with ∂V. The domain that the shell occupies is denoted by V_1. We assume the reinforcement and the massive body to be isotropic, to have different elastic moduli and be bonded together rigidly on ∂V. Within the framework of geometrically linear theory it is required to find the state of stress and strain of an inhomogeneous body $V_* = V_0 \cup V_1$ subjected to certain external forces applied to the body V_0.

Let F_i be external mass forces acting on the body V_0 and p^i surface forces given on a part of the boundary S_σ of the domain V_0: $\partial V_0 = S_\sigma \cup S_u$. The indices $i, j, k, \ldots$ take the values 1, 2, 3 and correspond to projections on the axes of a Cartesian x^i coordinate system. According to /1/, the solution of the problem in question should satisfy a system of elasticity theory equations and boundary conditions in the domains V_0 and V_1 and contact conditions on the surface $\Omega = \partial V$. They have the form:

in the domain V_0

$$\sigma^{ij}_{,j} + F^i = 0, \quad \sigma^{ij} = \lambda_0 \delta^{ij} (\delta^{kl} \varepsilon_{kl}) + 2\mu_0 \varepsilon^{ij} \tag{1.1}$$
$$\varepsilon_{ij} = (w_{i,j} + w_{j,i})/2, \quad \sigma^{ij} n_j |_{S_\sigma} = P^i, \quad w_i |_{S_u} = w_i^*$$

in the domain V_1

$$p^{ij}_{,j} = 0, \quad p^{ij} = \lambda_1 \delta^{ij} (\delta^{kl} e_{kl}) + 2\mu_1 e^{ij} \tag{1.2}$$
$$e_{ij} = (u_{i,j} + u_{j,i})/2, \quad p^{ij} n_j |_{\Omega_-} = 0$$

The contact conditions are expressed as follows:

$$w_i |_\Omega = u_i |_\Omega, \quad (\sigma^{ij} n_j - p^{ij} n_j) |_\Omega = 0 \tag{1.3}$$

Here w_i, u_i are, respectively, the displacement vector components of points of the body in the domains V_0 and V_1, ε_{ij}, e_{ij} are strain tensors, σ^{ij}, p^{ij} are stress tensors, λ_0, μ_0 are Lame elastic constants of the body V_0, λ_1, μ_1 are reinforcements of V_1, n_i are normal vectors to the appropriate surfaces S_σ, Ω and Ω_-, and Ω_- is the interior facial surface of the reinforcement. The last conditions in (1.1) mean that displacements are given on the part S_u of the boundary of the massive body. Kronecker deltas are denoted by δ^{ij} and a comma in the subscripts denotes the operation of partial differentiation.

The problem formulated is among the class of so-called contact problems between thin-walled elements and massive deformable bodies. The thin-walled reinforcement should obviously be taken into account in the investigation of such problems.

2. The transformed problem. Besides the original problem of the deformation of an inhomogeneous body $V_0 \cup V_1$ we consider the elasticity theory problem just for the domain V_0 (1.1) with a boundary condition on Ω of the form

$$\sigma_j^{\cdot i} r_\alpha^{\cdot j} n_i |_\Omega = -T^\beta_{\alpha;\beta} + b_{\alpha\mu} M^{\mu\sigma}_{;\sigma}, \quad r_\alpha^{\cdot i} = \partial r^i / \partial \xi^\alpha \tag{2.1}$$
$$\sigma^{ij} n_i n_j |_\Omega = -M^{\alpha\beta}_{;\alpha\beta} - b_{\alpha\beta} T^{\alpha\beta}$$

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where $x^i = r^i(\xi^\alpha)$ is the equation of the surface Ω, n_i is the normal vector to Ω and ξ^α are coordinates on the contact surface. The superscripts $\alpha, \beta, \gamma, \ldots$ take the values 1, 2 and correspond to projections on the coordinate axes ξ^α. We denote the tensor of the second quadratic form of the surface Ω by $b_{\alpha\beta}$ while the semicolon in the subscripts denotes the operation of covariant differentiation with respect to the connectedness on Ω. The tensors $T^{\alpha\beta}$ and $M^{\alpha\beta}$ are related to the displacements of points of the boundary of the boundary of the massive body by the formulas

$$T^{\alpha\beta} = \partial\Phi/\partial\gamma_{\alpha\beta}, \quad M^{\alpha\beta} = -\partial\Phi/\partial\rho_{\alpha\beta} \tag{2.2}$$

$$\Phi = \mu_1 h \{\gamma^{\alpha\beta}\gamma_{\alpha\beta} + \sigma(\gamma_\lambda^{\ \lambda})^2 - h[\gamma^{\alpha\beta}\rho_{\alpha\beta} + \sigma\gamma_\lambda^{\ \lambda}\rho_\lambda^{\ \lambda}] + {}^1/_3 h^2[\rho^{\alpha\beta}\rho_{\alpha\beta} + \sigma(\rho_\lambda^{\ \lambda})^2]\}, \quad \sigma = \lambda_1/(3\lambda_1 + 2\mu_1) \tag{2.3}$$

$$\gamma_{\alpha\beta} = u_{(\alpha;\beta)} - b_{\alpha\beta}u, \quad \rho_{\alpha\beta} = B_{\alpha\beta} - b^\lambda_{(\alpha}\gamma_{\lambda\beta)}, \quad u_\alpha = r_\alpha^{\ i}u_i, \quad u = u_i n^i$$

$$B_{\alpha\beta} = (u_{,\alpha} + b_\alpha^{\ \lambda}u_\lambda)_{;\beta} + b_\alpha^{\ \lambda}(u_{\lambda;\beta} - b_{\lambda\beta}u) \tag{2.4}$$

The tensors $\gamma_{\alpha\beta}$ and $\rho_{\alpha\beta}$ characterize the tension and bending of the contact surface Ω, and the parentheses in the subscript denote the tensor symmetrization operation: $A_{(\alpha\beta)} = (A_{\alpha\beta} + A_{\beta\alpha})/2$. The form of the boundary Conditions (2.1)-(2.4) allows of their physical interpretation as follows. As already noted $\gamma_{\alpha\beta}$ and $\rho_{\alpha\beta}$ are measures of the tension and bending of the surface Ω, the function Φ is the density of the elastic reinforcement energy referred to unit area of the contact surface. Relationships (2.2) are analogous to the equations of state of two-dimensional shell theory and the boundary Conditions (2.1) to the equilibrium equations. These express the continuity of the stress in the initial contact problem on Ω.

The solution of Problem (1.1), (2.1)-(2.4) is proposed for determining the state of stress and strain of the inhomogeneous elastic body $V_0 \cup V_1$ instead of the contact Problem (1.1)-(1.3).

The answer to the natural question of how much the solutions of these problems are different and in what cases, yields the following fundamental assertion.

Assertion. Let $\boldsymbol{\sigma}^0$ be a stress field that is the solution of the contact Problem (1.1)-(1.3), and let $\boldsymbol{\sigma}$ correspond to the Problem (1.1), (2.1)-(2.4) (here and henceforth let the bold-faced letters denote tensors of the second rank). Then the following inequality holds:

$$\|\boldsymbol{\sigma}^0 - \boldsymbol{\sigma}\|_{L_2(V_*)} \leqslant C\mu_1\varepsilon(|V_0|^{1/2} + |V_1|^{1/2})(h/l + h/R)^{1/2} \tag{2.5}$$

$$\varepsilon = \varepsilon_\gamma + \varepsilon_\rho, \quad \varepsilon_\gamma = \sup(\gamma_{\alpha\beta}\gamma^{\alpha\beta})^{1/2}, \quad \varepsilon_\rho = \sup h(\rho_{\alpha\beta}\rho^{\alpha\beta})^{1/2}$$

Here C is a constant independent of h, μ_1 is the reinforcement shear modulus, ε_γ is the scale of the tensile strain of the contact surface, ε_ρ is the bending strain scale, and $|V_t|$ is the volume of the domain V_t $(t = 0, 1)$. The operation **sup** is taken here and henceforth over the surface Ω. The characteristic radius of curvature R of the surface Ω is determined by the relationship $1/R = \sup(b_{\alpha\beta}b^{\alpha\beta})^{1/2}$. The characteristic scale of the change in deformation is introduced by the formula $l = \min(l_1, l_2)$, where l_1, l_2 are the best constants in the system of inequalities

$$\sup|\gamma_{\alpha\beta,\gamma}| \leqslant \varepsilon_\gamma/l_\nu, \quad \sup h|\rho_{\alpha\beta,\gamma}| \leqslant \varepsilon_\rho/l_\nu$$

The norm L_2 of the second rank tensor $\boldsymbol{\sigma}$ is assumed to be the following:

$$\|\boldsymbol{\sigma}\|_{L_2(V_t)} = \Big[\int_{V_t} \sigma^{ij}\sigma_{ij}d\tau_t\Big]^{1/2}$$

where $d\tau_t$ is a volume element of the domain V_t.

Inequality (2.5) determines the error in the solution of the transformation of the problem. If $R \gg h$ and the solution of the transformation of the problem is such that $h/l \ll 1$, then in the norm L_2 it differs from the solution of the original contact problem by terms of the order of $(h/l + h/R)^{1/2}$ as compared with the main terms. In fact the above assertion is equivalent to the following: as $h \to 0$ the solution of the transformation of the problem in the norm L_2 tends to the solution of the original contact problem; the asymptotic equivalence of the solutions of these problems is thereby established.

We will prove the above assertion.

3. *Error estimates*. The error in the solution of the transformation of the problem is estimated successfully because of the following modification of the Prager-Synge /2/ identity. We give the displacement

$$u^i|_\Omega = \varphi^i \tag{3.1}$$

on Ω in some manner and we examine two problems in addition to the original contact Problem (1.1)-(1.3): first is (1.1) and (3.1) and second is (1.2) and (3.1). We use the notation $E_{V_*}, E_{V_0}, E_{V_1}$, respectively, for the additional energy in the original, first, and second problems

$$E_{V_0} = \int\limits_{V_0} E_0^{-1ijkl}\sigma_{ij}\sigma_{kl}d\tau_0, \quad E_{V_1} = \int\limits_{V_1} E_1^{-1ijkl}\sigma_{ij}\sigma_{kl}d\tau_1, \quad E_{V_*} = E_{V_0} + E_{V_1}$$

where $E_0^{-sijkl}, E_1^{-sijkl}$ are elastic compliance tensors /3/, respectively, in the domains V_0 and V_1. Then

$$E_{V_*}[\sigma^\circ - (\sigma' + \bar{\sigma})/2] = {}^1\!/_4 E_{V_0}[\sigma - \bar{\sigma}] + {}^1\!/_4 E_{V_1}[\mathrm{p} - \bar{\sigma}] + \int\limits_{\Omega} (p^{ij} - \sigma^{ij}) n_j (w_i^\circ - \varphi_i)\, d\sigma, \ \sigma' = \{\sigma{:}x^i \in V_0,\ p{:}x^i \in V_1\} \tag{3.2}$$

Here σ°, w_i° is the solution of the original contact problem, σ, p is the statically allowable stress fields in the first and second problems, respectively, $\bar{\sigma}$ is the kinematically allowable field in the original problem consistent with (3.1), and $d\sigma$ is the area element of the surface Ω.

The difference between the identity (3.2) and the Prager-Synge identity is that σ' is not a statically allowable stress field in the original problem. It becomes such if the condition $(p^{ij} - \sigma^{ij}) n_j = 0$ is satisfied on Ω; then (3.2) is transformed into the Prager-Synge identity.

Since $E(\sigma)$ is a positive-definite quadratic form in the space of all possible states of stress of an elastic body, $E^{1/2}$ can be identified with the norm L_2. Then the stress field $(\sigma' + \sigma)/2$ can be considered as an approximation of the solution of the initial problem in the norm L_2 if the field $\bar{\sigma}, \sigma, \mathrm{p}$ is constructed successfully so that the right side of (3.2) is small.

Let us perform the procedure mentioned to estimate the error of the solution of the problem transformation.

Let $w_i^*, \sigma_{ij}^*, e_{ij}^*$ be the solution of the transformed problem. We give the displacement on Ω in the form (3.1) where $\varphi_i = w_i^*$. We select the solution of the transformed problem σ_{ij}^*: $\sigma_{ij} = \sigma_{ij}^*\ \bar{\sigma}_{ij} = \sigma_{ij}^*$ in the domain V_0 as the statically and kinematically allowable stress fields in the first problem. Then the first component on the right side of inequality (3.2) vanishes.

To construct the statically and kinematically allowable stress fields in Problem 2 we introduce a curvilinear coordinate system in the domain V_1 by means of the formulas

$$x^i(\xi^\alpha, \xi) = r^i(\xi^\alpha) + \xi n^i(\xi^\alpha) \tag{3.3}$$

By replacing the desired functions

$$s^{\alpha\beta} = (a_\nu{}^\beta - \xi b_\nu{}^\beta)\, p^{\alpha\nu}\varkappa, \quad s^{\alpha 3} = p^{\alpha 3}\varkappa, \quad \varkappa = 1 - 2\xi H + \xi^2 K \tag{3.4}$$

in (1.2) the equilibrium equation can be transformed /2/ into the form

$$s^{\alpha\beta}{}_{;\beta} - b_\beta{}^\alpha s^{\beta 3} + s^{3\alpha}{}_{,\xi} = 0, \quad s^{\alpha 3} - \xi b_\beta{}^\alpha s^{\beta 3} = s^{3\alpha}$$
$$s^{\alpha 3}{}_{;\alpha} + b_{\alpha\beta} s^{\alpha\beta} + s^{33}{}_{,\xi} = 0 \tag{3.5}$$

The subscripts $a, b, c, \ldots$ take the values 1, 2, 3 and correspond to projections on the coordinate axes $\{\xi^\alpha, \xi\}$, $a_{\alpha\beta}$ is the first quadratic form of the surface Ω, H is the mean and K is its Gaussian curvature. As before, the comma in the subscripts denotes partial differentiation, the semicolon is the covariant differentiation relative to the connectedness in Ω.

Specifying $s^{\alpha\beta}$ in some manner, (3.5) can be considered as a system of ordinary differential equations in $s^{\alpha 3}, s^{33}$. The boundary condition in (1.2) in terms of the substitution made for the desired functions has the form

$$\xi = -h, \quad s^{3\alpha} = 0, \quad s^{33} = 0 \tag{3.6}$$

Let us take $s^{\alpha\beta}$ in the form

$$s^{\alpha\beta} = h^{-1}(4T^{*\alpha\beta} + 6h^{-1}M^{*\alpha\beta}) + 12h^{-3}(M^{*\alpha\beta} + {}^1\!/_2 hT^{*\alpha\beta})\,\xi \tag{3.7}$$

It is clear that a solution of Problem (3.5) and (3.6) for $s^{\alpha 3}, s^{33}$ exists. Writing it explicitly, it can be seen that the desired functions are quantities of the order of $\mu_1 e\,(h/l + h/R)$. For $\xi = 0$ i.e., on the contact surface Ω, their value is obtained by integrating (3.5) with respect to ξ between $-h$ and 0. Since Conditions (3.6) are valid, we obtain

$$\xi = 0, \quad -s^{3\alpha} = -b_\beta{}^\alpha N^{*\beta} + T^{*\alpha\beta}{}_{;\beta}, \quad -s^{33} = N^{*\alpha}{}_{;\alpha} + b_{\alpha\beta}T^{*\alpha\beta}$$
$$N^{*\alpha} = M^{*\alpha\beta}{}_{;\beta}$$

The last relationship is obtained by integrating the first equation in (3.5) multiplied by ξ with respect to ξ between $-h$ and 0. This finally yields

$$\xi = 0, \quad s^{3\alpha} = -T^{*\alpha\beta}{}_{;\beta} + b_\nu{}^\alpha M^{*\nu\sigma}{}_{;\sigma}, \quad s^{33} = -M^{*\alpha\beta}{}_{;\alpha\beta} - b_{\alpha\beta}T^{*\alpha\beta} \tag{3.8}$$

We will construct the kinematically allowable stress fields in Problem 2 by using the following representation of the displacement field of points of the domain V_1

$$u^i(\xi^\alpha, \xi) = \varphi^i(\xi^\alpha) - \xi r_\nu{}^i a^{\nu\alpha} n_j \varphi^j{}_{,\alpha} + hy^i(\xi^\alpha, \xi) \tag{3.9}$$

This enables us to write the strain tensor components e_{ab} in the curvilinear system of coordinates $\{\xi^\alpha, \xi\}$ in the form

$$e_{\alpha\beta} = \gamma^*_{\alpha\beta} - \xi B^*_{\alpha\beta} + h(y_{(\alpha;\beta)} - b_{\alpha\beta}y) + hb^\nu_{(\alpha}(y_{\nu;\beta)} - b_{\nu\beta)}y) + \xi^2 b^\lambda_{(\alpha}B_{\lambda\beta)} \tag{3.10}$$
$$e_{\alpha3} = hy_{\alpha,\xi} + hy_{,\alpha} + hb_\alpha{}^\lambda y_\lambda - h^2 b_\alpha{}^\lambda y_{\lambda,\xi}, \quad e_{33} = hy_{,\xi}$$
$$y = y_i n^i, \quad y_\alpha = r_\alpha{}^i y_i$$

The kinematically allowable displacement field $\bar{s}^{ab}$ is evaluated according to (3.10) in conformity with Hooke's law and (3.4).

We take y_α, y in the form

$$y_\alpha \equiv 0, \quad y = -\sigma\gamma_\lambda^{*\lambda} + h^{-1}\sigma/2B_\lambda^{*\lambda}(\xi^2 - h^2/12) \tag{3.11}$$

Then $\bar{s}^{\alpha3}$ and $\bar{s}^{33}$, as follows from (3.10), will be quantities of the order of $\mu_1\varepsilon(h/l + h/R)$. By virtue of (3.10) and the equations of state (2.2) and (2.3), an analogous deduction holds for the difference $s^{\alpha\beta} - \bar{s}^{\alpha\beta}$. Therefore, the statically allowable field s^{ab} (3.7) and the kinematically allowable field $\bar{s}^{ab}$ corresponding to (3.9) and (3.11) between which the point by point difference is a quantity of the order of $\mu_1\varepsilon(h/l + h/R)$ are constructed in the domain V_1. This enables us to write

$$[E_{V_1}(p - \bar{\sigma})]^{1/2} \leqslant C'\mu_1\varepsilon |V_1|^{1/2}(h/l + h/R) \tag{3.12}$$

It remains to compare the values of the statically allowale fields σ and $\mathbf{p}$ on Ω. Since, as follows from (3.4),

$$p^{ij}n_i n_j = s_{33}, \quad p^{ij}n_j r_{i\alpha} = s_{3\alpha}$$

on Ω we obtain from (3.8) and the boundary Condition (2.1) for the transformed problem that $(\sigma^{ij} - p^{ij})n_j = 0$. The third component on the right side of the identity (3.2) thereby vanishes. We therefore obtain

$$\|\sigma^\circ - (\sigma' + \bar{\sigma})/2\|_{L_2(V_*)} \leqslant C\mu_1\varepsilon |V_1|^{1/2}(h/l + h/R) \tag{3.13}$$

The relationship $|V_8|^{1/2} \sim hL^{-1}|V_0|^{1/2}$ holds, where L is the characterstic dimension of the cavity and $L \gg R$, $L \gg l$. Then (2.5) follows trivially from (3.13).

Remarks. 1°. It was noted in Sect.1 that the problem considered is among the class of contact problems of massive deformable bodies with thin-walled elements. An extensive literature is devoted to it, whose bibliography is represented fairly completely in /3, 4/. The idea of using the thin-walledness of the reinforcing elements in the plan for their approximate analysis by the theory of rods and shells is not new. It has repeatedly been applied in both problems of the theory of shells with stiffener ribs, with reinforced edges and holes, /5-7/, say, and in problems on the contact of massive deformable bodies with thin-walled elements /3, 4, 8/. General considerations on a variational approach to this problem are given in /9/.

2°. The central theme of this paper is to obtain error estimates of the solution of the transformed problem as compared with the original contact problem. It is important here to distinguish the error of the solution from the errors of the equations of the transformed problem themselves. The latter is defined as the relative magnitude of the discarded small terms in relationships (1.1)-(1.3) during passage over to Problem (1.1), (2.1)-(2.4). Namely, the errors of the equations are discussed in /3/ when the level of the accuracy of the applied equations for thin coatings is discussed. The modification (3.2) of the Prager-Synge identity was obtained* (*Misyura, V.A., Effect of the loss of accuracy of classical shell theory. Candidate Dissert., Moscow State Univ., 1983.) in connection with giving a foundation for the fundamental hypotheses and relationships of two-dimensional shell theory.

3°. It can be verified that the transformed problem allows of a variational formulation. The Lagrange functional for it has the form

$$I(w^i) = \int_{V_0} U d\tau_0 - \int_{S_\sigma} P_i w^i d\sigma - \int_{V_0} F_i w^i d\tau_0 + \int_{\Omega} \Phi d\omega$$

where U is the elastic energy density of the massive body, $d\tau_0$ is the volume element of the domain V_0 and $d\sigma, d\omega$ are the area elements of the surfaces S_σ and Ω, respectively. This enables us to use the well-developed variational-difference method for a numerical investigation of a broad class of problems.

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ON A METHOD OF INVESTIGATING FIBRE STABILITY IN AN ELASTIC SEMI-INFINITE MATRIX NEAR A FREE SURFACE*

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The properties of an infinite characteristic determinant in the three-dimensional linearized problem /1/ of fibre stability in an elastic semi-infinite matrix near a free surface are investigated. As in the case of two and a number of doubly-periodic systems of fibres in an infinite matrix /2-4/ it is proved that the mentioned determinant is a determinant of normal type. Non-linearly elastic transversely isotropic compressible materials are examined within the framework of the theory of finite subcritical deformations without taking account of the specific form of the elastic potential. The results elucidated hold even for different modifications of the theory of small subcritical deformations. Results of an analysis of kindred questions of the theory of elastic wave diffraction are used /5/.

1. ***Formulation of the problem.*** The characteristic equation. We consider the stability

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